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Sections of the Taylor Expansions of Lindelöf Functions*

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DEDICATED TO THE MEMORY OF GÉZA FREUD

INTRODUCTION

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1}$$

be an entire function of order λ $(0 < \lambda < +\infty)$.

We propose to obtain precise information regarding the distribution of zeros of the partial sums

$$s_{m-1}(z) = \sum_{k=0}^{m-1} a_k z^k, \tag{2}$$

as $m \to +\infty$ through values of a suitable sequence. Since the series in (1) may have gaps, it is not certain that $s_{m-1}(z)$ is a polynomial of exact degree $m-1$. Hence we introduce the exact degree of the partial sum $s_{m-1}(z)$ and denote it by $d(m-1)$. All the functions treated in this paper are simple enough to imply

$$d(m-1) \sim m \qquad (m \to +\infty).$$

The problem of studying the angular distribution of the zeros of $s_{m-1}(z)$, as $m \to +\infty$, is not new. In first approximaton it was solved by Carlson [1, 2] and Rosenbloom, in his remarkable thesis [13, 14].

Our aim is to obtain theorems similar to those of Carlson and Rosen-

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bloom, but considerably sharper. Completely satisfactory results along such lines are known for some isolated functions, notably for e^z [17, 15], for

$$\int_0^z e^{-\zeta^2}\,d\zeta, \tag{3}$$

and for the functions of Mittag-Leffler of all orders λ $(0<\lambda<+\infty)$:

$$E_{1/\lambda}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(1+j/\lambda)}. \tag{4}$$

The study of e^z was undertaken by Szegö [17]. His penetrating analysis still dominates the subject and seems to have inspired a good deal of subsequent research [14, 15, 4]. The study of the error function (3) is due to Rosenbloom [13], and the study of Mittag-Leffler's function to Edrei, Saff, and Varga [7].

The functions in (3) and (4) display some common features:

(i) the coefficients of their Taylor expansions are explicitly known;

(ii) as $z\to\infty$, the asymptotic behavior of $f(z)$ is simple and easy to describe.

The first of these properties is not directly used in our proofs whereas property (ii) plays a dominant role.

This explains the fact that, from the point of view of this paper, the function $1/\Gamma(z)$ may be treated satisfactorily without requiring some preliminary study of its rather awkward Taylor expansion.

The function

$$e^{-Cz}/z\Gamma(z)=\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-(z/j)} \qquad (C=\text{Euler's constant}) \tag{5}$$

belongs to a class of canonical products introduced and studied by Lindelöf [10]. The remarkable asymptotic behavior of these products has inspired many conjectures (and proofs) regarding the more delicate aspects of the theory of entire and meromorphic functions. In particular, Wiman's cos $\pi\rho$-theorem [20], the tauberian results of Valiron [19], and some of the deficiency problems of Nevanlinna's theory [11, 3, 6] were conjectured on the strength of Lindelöf's asymptotic evaluations.

The Lindelöf functions are defined as follows:

(i) they are canonical products of the form

$$\mathscr{L}(z)=\prod_{j=1}^{\infty}E\left(-\frac{z}{x_j},q\right) \qquad (q\geqslant 0), \tag{6}$$

where

$$E(u, q) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^q}{q}\right), \qquad E(u, 0) = 1-u,$$

is the primary factor of genus q;

(ii) the quantities x_j are all real and positive and the counting function of the zeros of $\mathscr{L}(z)$,

$$n(t) = \sum_{x_j \leqslant t} 1,$$

satisfies the asymptotic relation

$$n(t) \sim b_0 t^\lambda (\log t)^\alpha \qquad (t \to +\infty, b_0 > 0, 0 < \lambda < +\infty, \alpha \text{ real}); \quad (7)$$

(iii) by definition, $q \geqslant 0$ is the least nonnegative integer such that

$$\sum_j x_j^{-q-1} < +\infty$$

and hence by (7), $q = [\lambda]$ if λ is not an integer. If λ is an integer, (7) implies

$$q = \lambda - 1 \ (\alpha < -1), \qquad q = \lambda \ (\alpha \geqslant -1). \quad (8)$$

Our main results, Theorems 1, 2, and 3 below, will enable us to describe, with satisfactory precision, the zeros of the partial sums $s_{m-1}(z)$ of the Taylor expansion of any one of the Lindelöf functions described above. A brief sketch of our analysis of this question is found in Section 1B. It will often prove possible to completely treat functions of the form

$$\tilde{\mathscr{L}}(z) = \prod_{k=1}^{K} \mathscr{L}(B_k e^{i\theta_k} z) \qquad (B_k > 0, \theta_k \text{ real}, 1 \leqslant k \leqslant K), \quad (9)$$

where $\mathscr{L}$ denotes the same Lindelöf function in all K factors of the right-hand side. For functions of the more complex form $\tilde{\mathscr{L}}$ some caution should be exercised. In order to illustrate the slight additional complications which may arise in the study of (9) we write

$$L(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{x_j}\right)$$

and assume that all the x_j are positive and that their counting function $n(t)$ satisfies

$$n(t) \sim t^\lambda \qquad (t \to +\infty, \tfrac{1}{2} < \lambda < 1).$$

We then introduce

$$\mathscr{L}_1(z) = L\left(-z \exp\left(i\frac{\pi}{2\lambda}\right)\right) L\left(-z \exp\left(-i\frac{\pi}{2\lambda}\right)\right), \tag{10}$$

and, in Section 1C, study the sections of the power series expansion of $\mathscr{L}_1(z)$.

Our proofs depend on the simplest aspects of the Wiman–Valiron theory of the maximum term. We require no knowledge beyond what may be found in [12, 16].

Our first result, stated below as Theorem 1, is little more than a lemma. It establishes, in the exact form in which we use them, the terminology and notations regarding the maximum term and the central index. The theorem also defines an infinite sequence

$$\mathfrak{M}: m_1, m_2, m_3, \ldots, \tag{11}$$

of nonnegative, strictly increasing integers. The sequence $\mathfrak{M}$ is fundamental in our treatment, because in all our statements about $s_{m-1}(z)$, we always require $m \in \mathfrak{M}$. For ease of reference we shall refer to $\mathfrak{M}$ as the *sequence of critical indices.*

THEOREM 1. *Let $f(z)$ be an entire, transcendental function given by its expansion* (1). *Let*

$$\mu(r) = \max_j |a_j| \, r^j \qquad (r > 0)$$

be its maximum term and $\nu = \nu(r)$ its central index, that is, the integer $\nu \geq 0$ characterized by the conditions

$$|a_\nu| \, r^\nu = \mu(r), \qquad \left|\frac{a_{\nu+j}}{a_\nu}\right| r^j < 1 \qquad (j > 0). \tag{12}$$

Then, if

$$M(r, f) = M(r) = \max_{|z| = r} |f(z)|, \tag{13}$$

satisfies the asymptotic relation

$$\log M(r) \sim Br^\lambda (\log r)^\alpha \quad (r \to \infty, 0 < \lambda < +\infty, B > 0, \alpha \text{ real}), \tag{14}$$

we have

$$\nu(r) \sim \lambda B r^\lambda (\log r)^\alpha. \tag{15}$$

Denote by $\mathfrak{M}$ *the sequence formed by taking, in their natural order, all those integers* $\geqslant 0$, *which are actual values of the central index.*

The sequence $\mathfrak{M}$ *is the sequence of critical indices introduced in* (11).

It is always possible to associate with $\mathfrak{M}$ *a positive, strictly increasing, unbounded sequence of radii*

$$\{R_{m_j}\}_{j=1}^{\infty}, \tag{16}$$

such that

$$\nu(R_m) = m \sim \lambda B R_m^{\lambda} (\log R_m)^{\alpha} \qquad (m \to \infty, m \in \mathfrak{M}). \tag{17}$$

Moreover if $d(m-1)$ *denotes the largest integer such that*

$$d = d(m-1) < m, \qquad a_d \neq 0, \tag{18}$$

then

$$d(m-1) \sim m \qquad (m \to \infty, m \in \mathfrak{M}). \tag{19}$$

Define

$$a_{-k} = 0 \qquad (k = 1, 2, 3, ...), \tag{20}$$

and set

$$b_j(m) = \frac{a_{m+j}}{a_m} R_m^j \qquad (m \in \mathfrak{M}, j = 0, \pm 1, \pm 2, ...). \tag{21}$$

For all such $b_j(m)$ *we have*

$$|b_j(m)| \leqslant 1. \tag{22}$$

Remark. The particular form of our asymptotic condition (14) has been chosen for simplicity. The insertion in the right-hand side of (14) of finitely many factors such as

$$(\log \log r)^{\alpha_1}, (\log \log \log r)^{\alpha_2}, ...,$$

introduces minimal changes in our proofs but presents the drawback of complicating the writing. The corresponding extension of our next results (Theorems 2 and 3) is possible and straightforward.

COROLLARY 1.1. *Let* w *be an auxiliary complex variable. For the entire functions of* w

$$G_m(w) = \sum_{j=0}^{\infty} b_j(m) w^j \qquad (m \in \mathfrak{M}), \tag{23}$$

we have

$$G_m(0)=1, \qquad |G_m(w)| \leqslant \frac{1}{1-|w|} \qquad (|w|<1). \tag{24}$$

From every infinite subsequence $\mathfrak{M}_1 \subset \mathfrak{M}$, *it is possible to extract an infinite subsequence* $\mathfrak{M}_2$ *such that*

$$b_j(m) \to b_j \qquad (m \to +\infty, m \in \mathfrak{M}_2, j=0, \pm 1, \pm 2,...). \tag{25}$$

Introducing

$$G(w)=\sum_{j=0}^{\infty} b_j w^j \qquad (|w|<1, G(0)=1),$$

we have

$$G_m(w) \to G(w) \qquad (m \to \infty, m \in \mathfrak{M}_2),$$

uniformly on every compact subset of the disk $|w|<1$.

THEOREM 2. *Let the assumptions and notations of Theorem* 1 *be unchanged. Assume in addition that*

$$\log f(z)=\beta z^{\lambda}(\log z)^{\alpha}+o(|z|^{\lambda}(\log |z|)^{\alpha}), \tag{26}$$

uniformly, as $z \to \infty$ *in some angle*

$$\varphi_1 \leqslant \arg z=\varphi \leqslant \varphi_2 \qquad (\varphi_1<\varphi_2<\varphi_1+2\pi). \tag{27}$$

Restrict φ *by the condition*

$$\varphi_1+\gamma \leqslant \varphi \leqslant \varphi_2-\gamma \qquad (0<\gamma<\tfrac{1}{4}(\varphi_2-\varphi_1)) \tag{28}$$

and assume that, for such values of φ

$$1-\operatorname{Re}\frac{\beta}{B} e^{i\varphi\lambda}=c=c(\varphi) \geqslant c_0>0. \tag{29}$$

I. *Then the equation in* x

$$(1-c)\, x^{\lambda}-1-\lambda \log x=0 \qquad (c=c(\varphi)) \tag{30}$$

has a unique solution $x=\tau(\varphi)$ *in the interval* (0, 1).

More precisely

$$\exp\left(-\frac{2+(|\beta|/B)}{\lambda}\right)=\tau_0<\tau(\varphi)<\tau_1=\exp\left(-\frac{\min(1, c_0)}{2\lambda+1}\right), \tag{31}$$

and all the following assertions are valid.

II. *There exists a positive sequence* $\{t(m)\}$ $(m \in \mathfrak{M})$, *such that*

$$t(m) \to \tau = \tau(\varphi) \qquad (m \to \infty, m \in \mathfrak{M}) \tag{32}$$

and such that the polynomials in ζ *defined by*

$$X_m(\zeta) = (a_m R^m)^{-1} \{t(m)\}^{-m} e^{-i\varphi m} s_{m-1}\left(Rt(m)\, e^{i\varphi}\left(1 + \frac{\zeta}{\Lambda m}\right)\right) \tag{33}$$

with

$$m \in \mathfrak{M}, \qquad R = R_m, \qquad \Lambda = \frac{\beta}{B}\xi^{\lambda} - 1 \neq 0, \qquad \xi = \xi(\varphi) = \tau(\varphi)\, e^{i\varphi} \tag{34}$$

are uniformly bounded on every compact set of the ζ*-plane.*

The exact degree $d(m-1)$ *of* $s_{m-1}(z)$ *(or of* $X_m(\zeta)$*) satisfies the condition*

$$d(m-1) \sim m \qquad (m \to \infty, m \in \mathfrak{M}). \tag{35}$$

III. *From every infinite subsequence* $\mathfrak{M}_1 \subset \mathfrak{M}$ *it is possible to extract an infinite subsequence* $\mathfrak{M}_2 \subset \mathfrak{M}_1$ *such that, with the notations of Corollary* 1.1,

$$G_m(w) \to G(w) \qquad (m \to \infty, m \in \mathfrak{M}_2, G(0)), \tag{36}$$

uniformly on every compact subset of the disk $|w| < 1$.

Considering if necessary an infinite subsequence $\mathfrak{M}_3 \subset \mathfrak{M}_2$ *we have*

$$X_m(\zeta) \to e^{\zeta/\Lambda}(e^{i\chi}e^{\zeta} - G(\xi)) = \tilde{X}(\zeta) \qquad (m \to \infty, m \in \mathfrak{M}_3), \tag{37}$$

uniformly on every compact set of the ζ*-plane.*

The quantity χ *which appears in* (37) *is real; it may depend on the choice of* $\mathfrak{M}_3$.

It is clear that the preceding assertions II and III (with the exception of (35)) could be replaced by

IV. *The family of polynomials* $\{X_m(\zeta)\}$ $(m \in \mathfrak{M})$ *is normal throughout the* ζ*-plane and every one of its limit-functions is of the form indicated in the right-hand side of* (37).

Our aim is to derive from (37) the fact that $s_{m-1}(z)$ has many zeros in the immediate vicinity of the point

$$R_m t(m)\, e^{i\varphi}.$$

This requires that

$$G(\xi) \neq 0 \qquad (\xi = \tau(\varphi)\, e^{i\varphi}). \tag{38}$$

Since $G(w)$ is a limit function of the functions $G_m(w)$ considered in (24), the following corollary refers to $G(w)$ associated with some specific choice of the infinite sequence $\mathfrak{M}_2$:

$$G(w) = \lim_{\substack{m \to \infty \\ m \in \mathfrak{M}_2}} G_m(w).$$

A simple way to ensure that (38) is satisfied is to select φ different from the arguments of the zeros of $G(w)$. We are thus led to

COROLLARY 2.1. *Let the assumptions and notations of Theorems* 1 *and* 2 *be unchanged. It is also assumed that the infinite sequence* $\mathfrak{M}_3 \subset \mathfrak{M}$, *and therefore also the limit function* $G(w)$, *has been selected. Assume, in addition, that* φ *does not coincide with any of the arguments of the finite number of zeros of* $G(w)$ *in*

$$|w| \leqslant \frac{1+\tau_1}{2}. \tag{39}$$

Denote by $\mathcal{N}_m(r)$ *the number of zeros of* $s_{m-1}(z)$ *in the disk*

$$|z - R_m t(m)\, e^{i\varphi}| \leqslant r \frac{R_m t(m)}{|A|\, m} \qquad (r > 0). \tag{40}$$

Then

$$\left| \mathcal{N}_m(r) - \frac{r}{\pi} \right| \tag{41}$$

remains bounded for all

$$r > 0, \qquad m > m_0, \qquad m \in \mathfrak{M}_3 \subset \mathfrak{M}_2. \tag{42}$$

All the zeros under consideration are simple.

At the cost of a slight loss of precision, it is possible to formulate a result analogous to Corollary 2.1, free from any exclusions regarding exceptional values of φ. We prove

THEOREM 3. *Let the assumptions and notations of Theorem* 2 *be unchanged, let* $\varphi \in [\varphi_1 - \gamma, \varphi_2 + \gamma]$, *and assume that the infinite sequence* $\mathfrak{M}_3 \subset \mathfrak{M}$, *and therefore also the limit function* $G(w)$, *has been selected. Then there exist positive constants* T, T_1, *independent of the choice of* φ, *such that the section* $s_{m-1}(z)$ *has, in the disk*

$$|z - R_m t(m)\, e^{i\varphi}| \leqslant T R_m t(m) \frac{\log m}{|A|\, m} \qquad (m > m_0,\, m \in \mathfrak{M}_3), \tag{43}$$

at least

$$N_m \geqslant T_1 \log m \tag{44}$$

zeros.

The proof of Theorem 3 depends on the following elementary Lemma A. The reader will find a simple proof of this lemma in [5].

LEMMA A. *Let $\phi(\zeta)$ be a nonconstant function, regular for $|\zeta| \leqslant 1$. Let ζ_0 be a point such that*

$$|\zeta_0| < 1, \qquad |\phi(\zeta_0)| \geqslant 1, \tag{45}$$

and let σ_1, σ_2, and h be positive quantities such that

$$|\zeta_0| < |\zeta_0| + \sigma_1 < \sigma_2 < 1, \qquad 0 < h < 1. \tag{46}$$

Put

$$H_1 = \max_{|\zeta|=1} \log |\phi(\zeta)|, \qquad H_0 = \max_\theta \log |\phi(\zeta_0 + \sigma_1 e^{i\theta})|, \tag{47}$$

and let

$$H_2 = \max_{|\zeta| = \sigma_2} \log |\phi(\zeta)| \geqslant hH_1. \tag{48}$$

Denote by $\mathcal{N}$ the number of zeros of $\phi(\zeta)$ in the disk $|\zeta| \leqslant 1$ and let

$$\sigma = \left\{\frac{h\sigma_1(1-\sigma_2)^2}{50}\right\}^{1/h_1}, \qquad h_1 = \frac{\log(1+\frac{1}{4}(1-\sigma_2)^2)}{\log(4/\sigma_1)}. \tag{49}$$

Then, if

$$H_0 \leqslant \sigma H_1, \tag{50}$$

we have

$$\mathcal{N} \geqslant \frac{\sigma H_1}{\log(4/\sigma_1)}. \tag{51}$$

1. APPLICATIONS AND DISCUSSION OF THE PRECEDING RESULTS

There are a number of well-known functions of analysis to which the preceding theorems are immediately applicable and lead to a satisfactory picture of the zeros of the sections of their expansions.

A. *The Function* $1/\Gamma(z)$

Let (1) be the expansion of

$$f(z)=\frac{1}{\Gamma(z)}=ze^{Cz}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-z/j} \qquad (C=\text{Euler's constant}).$$

The convergence $s_m(z)\to f(z)$, which is uniform on every compact region, shows that the number of negative zeros of $s_m(z)$ tends to infinity as $m\to+\infty$.

It is well known [11; p. 232] that

$$\log(1/\Gamma(z))\sim -z\log z, \tag{1.1}$$

uniformly in the angle

$$|\arg z|\leqslant \pi-\delta \qquad (0<\delta<\pi).$$

To cover the angle

$$\pi/2\leqslant \arg z\leqslant \pi,$$

we may combine (1.1) with the functional equation

$$f(z)=\frac{\sin \pi z}{\pi}\,\Gamma(1-z)$$

to ascertain that [9; p. 27]

$$\log M(r)\sim r\log r \qquad (r\to+\infty).$$

Hence Theorems 1, 2, and 3 are applicable to $f(z)=1/\Gamma(z)$; the values of the relevant parameters are

$$B=1, \qquad \beta=-1, \qquad \beta\cos\varphi=-\cos\varphi=1-c. \tag{1.2}$$

Our conclusions regarding the partial sums $s_{m-1}(z)$ of the expansion of $1/\Gamma(z)$ may be summarized as follows.

I. Theorem 1 guarantees the existence of an infinite sequence $\mathfrak{M}$ of critical degrees and an associated sequence $\{R_m\}$ ($m\in\mathfrak{M}$) with the following properties:

(i) $m\sim R_m\log R_m$ ($m\to+\infty$, $m\in\mathfrak{M}$), or equivalently

$$R_m\sim\frac{m}{\log m} \qquad (m\to+\infty, m\in\mathfrak{M}); \tag{1.3}$$

(ii) as $m\to+\infty$, $m\in\mathfrak{M}$, Corollary 1 is valid.

The only useful information provided by the corollary is the fact that $G(w) \not\equiv 0$.

II. Theorem 2 may be applied with

$$\varphi_1 = -\pi + \frac{\gamma}{2}, \qquad \varphi_2 = \pi - \frac{\gamma}{2}, \qquad 0 < \gamma < \frac{\pi}{2}. \tag{1.4}$$

In view of (1.2), the equation (30) takes the form

$$-x \cos \varphi - 1 - \log x = 0 \qquad (-\pi < \varphi < \pi). \tag{1.5}$$

This equation has a unique solution $x = \tau(\varphi)$ in the interval (0, 1) and the point

$$\xi(\varphi) = \tau(\varphi)\, e^{i\varphi} \tag{1.6}$$

describes a curve, in the ζ-plane, which Edrei, Saff, and Varga [7] call the *normalized Szegö curve of the sequence* $\{s_{m-1}(z)\}_{m \in \mathfrak{M}}$.

It may be of interest to note that the Szegö curve just defined is the reflexion, in the imaginary axis, of the Szegö curve obtained by Szegö in his study of $\exp(z)$.

III. From Theorem 3 we deduce:

Given $\varphi \in [-\pi + \gamma, \pi - \gamma]$ $(0 < \gamma < \pi/2)$ it is possible to find a sequence

$$\{t(m)\}_{m \in \mathfrak{M}_3} \qquad (0 < t(m) < 1),$$

such that

$$t(m) \to \tau(\varphi) \qquad (m \to +\infty, m \in \mathfrak{M}_3 \subset \mathfrak{M}),$$

and such that $s_{m-1}(z)$ has, in the disk defined by (43), no fewer than $T_1 \log m$ zeros. The positive constants T, T_1 depend on the choice of $\mathfrak{M}_3$ and on the choice of γ in (1.4); $\tau(\varphi)$ is the quantity in (1.6), $x = \tau(\varphi)$ satisfies the equation (1.5), R_m satisfies (1.3) and $|A| = |1 + \xi(\varphi)| > 1 - \cos \gamma$.

IV. Regarding the negative zeros of $s_m(z)$, we know a priori that as $m \to +\infty$, the number N_m^* of negative zeros of $s_m(z)$ will tend to infinity as $m \to +\infty$. This is an obvious consequence of the fact that, as $m \to +\infty$, $s_m(z) \to (1/\Gamma(z))$, uniformly on every compact subset of the z-plane.

B. *Lindelöf Functions of Non-integral Order* λ $(1 < \lambda < +\infty)$

We focus our attention on values of $\lambda \neq$ integer. If λ is an integer, some asymptotic formulae take a different form; there are no additional difficulties so that a detailed treatment may be left to the reader.

We start from $\mathscr{L}(z)$ defined in (6) and assume

$$1 \leqslant q < \lambda < q+1, \qquad q \text{ is an integer.} \tag{1.7}$$

With regard to the counting function $n(t)$ of the zeros of $\mathscr{L}(z)$ (by assumption they are all real and negative), we assume

$$n(t) \sim t^{\lambda} \qquad (t \to +\infty). \tag{1.8}$$

Putting

$$M(r) = \max_{|z|=r} |\mathscr{L}(z)|, \tag{1.9}$$

and using a well-known result [11; p. 227]

$$\log M(r) < Kr^{\lambda} \qquad (r > r_0), \tag{1.10}$$

we find

$$\limsup_{r \to +\infty} \frac{\log |\mathscr{L}(-r)|}{r^{\lambda}} < +\infty. \tag{1.11}$$

Lindelöf's classical result states [11; p. 232] that

$$\log \mathscr{L}(z) = (-1)^q \frac{\pi}{|\sin \pi\lambda|} z^{\lambda} + o(|z|^{\lambda}), \tag{1.12}$$

uniformly as $z \to +\infty$ in the sector

$$-\pi + \delta < \arg z = \theta < \pi - \delta \qquad (0 < \delta,\ \delta \text{ fixed}). \tag{1.13}$$

Hence, for θ fixed $(-\pi < \theta < \pi)$, we have

$$\lim_{r \to +\infty} \frac{\log |\mathscr{L}(re^{i\theta})|}{r^{\lambda}} = \frac{(-1)^q \pi}{|\sin \pi\lambda|} \cos(\lambda\theta). \tag{1.14}$$

In view of (1.10) and (1.14), the Phragmén–Lindelöf indicator of $\mathscr{L}(z)$ is everywhere finite and consequently it is continuous [9; p. 54]. We thus conclude that (1.11) takes the sharper form

$$\limsup_{r \to +\infty} \frac{\log |\mathscr{L}(-r)|}{r^{\lambda}} = (-1)^q \frac{\pi}{|\sin \pi\lambda|} \cos(\lambda\pi). \tag{1.15}$$

From (1.14) and (1.15) it follows that

$$\limsup_{r \to +\infty} \frac{\log M(r)}{r^{\lambda}} \leqslant \frac{\pi}{|\sin \pi\lambda|}, \tag{1.16}$$

and for q even, (1.14) with $\theta = 0$ yields

$$\liminf_{r \to +\infty} \frac{\log M(r)}{r^\lambda} \geqslant \frac{\pi}{|\sin \pi\lambda|}. \tag{1.17}$$

For q odd (1.17) is obtained by setting $\theta = \pi/\lambda$ in (1.14). Hence (1.16) and (1.17) imply, without case distinction,

$$\log M(r) \sim \frac{\pi r^\lambda}{|\sin \pi\lambda|} \qquad (r \to +\infty). \tag{1.18}$$

We thus see that Theorem 1 is applicable to $\mathscr{L}(z)$ with

$$B = \frac{\pi}{|\sin \pi\lambda|}.$$

In view of (1.12) Theorem 2 is also applicable with

$$\beta = (-1)^q \frac{\pi}{|\sin \pi\lambda|}.$$

The condition (29) takes the form

$$1 - (-1)^q \cos(\varphi\lambda) = c > 0.$$

There are exceptional values of φ which our Theorem 2 does not enable us to treat. For q even they are

$$\varphi = 0, \pm\frac{2\pi}{\lambda}, \pm\frac{4\pi}{\lambda}, \pm\frac{6\pi}{\lambda}, \ldots \qquad (|\varphi| < \pi), \tag{1.19}$$

and for q odd,

$$\varphi = \pm\frac{\pi}{\lambda}, \pm\frac{3\pi}{\lambda}, \ldots \qquad (|\varphi| < \pi). \tag{1.20}$$

It should be remarked that our function $\mathscr{L}(z)$ is not known with great precision since the only available information regarding the moduli of its zeros if given by the asymptotic relation (1.8).

It would be possible to assume more about the behavior of $n(t)$ but this would reduce the generality of the results. It appears more interesting to eliminate the exceptional values in (1.19) and (1.20) by following a pattern similar to the one used in the proof of our Theorem 3.

C. *Study of* $\mathscr{L}_1(z)$ *Given by* (10)

As shown in the study of $\mathscr{L}(z)$, the Phragmén–Lindelöf indicator of $\mathscr{L}_1(z)$ is a continuous function of $\theta = \arg z$.

The asymptotic relations of Lindelöf yield as $r \to +\infty$,

$$\log L\left(-z \exp\left(i\frac{\pi}{2\lambda}\right)\right) = \frac{\pi}{\sin \pi\lambda} r^{\lambda} \exp\left(i\left\{-\pi + \frac{\pi}{2\lambda} + \theta\right\}\lambda\right) + o(r^{\lambda})$$

$$\left(-\frac{\pi}{2\lambda} < \theta < 2\pi - \frac{\pi}{2\lambda}\right), \quad (1.21)$$

$$\log L\left(-z \exp\left(-i\frac{\pi}{2\lambda}\right)\right) = \frac{\pi}{\sin \pi\lambda} r^{\lambda} \exp\left(i\left\{-\pi - \frac{\pi}{2\lambda} + \theta\right\}\lambda\right) + o(r^{\lambda})$$

$$\left(\frac{\pi}{2\lambda} < \theta < 2\pi + \frac{\pi}{2\lambda}\right). \quad (1.22)$$

For

$$\frac{\pi}{2\lambda} < \theta < 2\pi - \frac{\pi}{2\lambda} \qquad \left(\frac{1}{2} < \lambda < 1\right), \tag{1.23}$$

the relations (1.21) and (1.22) are both valid and consequently

$$\log \mathscr{L}_1(z) = o(r^{\lambda}) \qquad (r \to +\infty, \theta = \arg z), \tag{1.24}$$

uniformly in θ, provided θ is restricted by

$$\frac{\pi}{2\lambda} + \delta \leqslant \theta \leqslant 2\pi - \frac{\pi}{2\lambda} - \delta \qquad \left(0 < \delta < \pi - \frac{\pi}{2\lambda}\right), \tag{1.25}$$

with δ fixed.

To evaluate $\mathscr{L}_1(z)$ outside the interval (1.23) we use, instead of (1.22), the equivalent form

$$\log L\left(-z \exp\left(-i\frac{\pi}{2\lambda}\right)\right) = \frac{\pi}{\sin \pi\lambda} r^{\lambda} \exp\left(i\left\{\pi - \frac{\pi}{2\lambda} + \theta\right\}\lambda\right) + o(r^{\lambda})$$

$$\left(-2\pi + \frac{\pi}{2\lambda} < \theta < \frac{\pi}{2\lambda}\right). \quad (1.26)$$

Combining (1.21) and (1.26) we see that as $r \to +\infty$ and for

$$-\frac{\pi}{2\lambda} + \delta \leqslant \theta \leqslant \frac{\pi}{2\lambda} - \delta \qquad \left(0 < \delta < \frac{\pi}{2\lambda}\right), \tag{1.27}$$

we have, uniformly in θ,

$$\log \mathscr{L}_1(z) = 2\pi r^{\lambda} e^{i\lambda\theta} + o(r^{\lambda}). \tag{1.28}$$

Hence

$$\max_{|z|=r} \log |\mathscr{L}_1(z)| \sim 2\pi r^{\lambda} \qquad (r \to +\infty). \tag{1.29}$$

The preceding relations yield the explicit form of the Phragmén–Lindelöf indicator of $\mathscr{L}_1(z)$. We find

$$\lim_{r \to +\infty} \frac{\log |\mathscr{L}_1(re^{i\theta})|}{2\pi r^{\lambda}} = \lim_{r \to +\infty} \frac{\log |\mathscr{L}_1(re^{i\theta})|}{\log M(r)} = H(\theta),$$

where

$$H(\theta) = \cos(\lambda\theta) \qquad \left(-\frac{\pi}{2\lambda} \leqslant \theta \leqslant \frac{\pi}{2\lambda}\right), \tag{1.30}$$

and

$$H(\theta) = 0 \qquad \left(\frac{\pi}{2\lambda} \leqslant \theta \leqslant 2\pi - \frac{\pi}{2\lambda}\right). \tag{1.31}$$

From (1.31) we conclude that the normalized Szegö curve associated with $\mathscr{L}_1(z)$ (its equation is given by (30)) contains the circular arc

$$e^{-1/\lambda} e^{i\varphi} \qquad \left(\frac{\pi}{2\lambda} \leqslant \varphi \leqslant 2\pi - \frac{\pi}{2\lambda}\right).$$

In view of (1.24), (1.28), and (1.29), Theorems 1 and 2 may be applied to $\mathscr{L}_1(z)$. The exceptional values of φ, not covered by Theorem 2, are

$$\varphi = 0, \ \pm\pi/2\lambda.$$

D. *The Function* $\exp(p(z))$ *Where* $p(z)$ *Is a Polynomial*

We take

$$p(z) = z^l + b_1 z^{l-1} + b_2 z^{l-2} + \cdots + b_l,$$

where the b_j are complex constants. It is clear that

$$\log f(z) \sim z^l \qquad (z \to \infty, f(z) = \exp(p(z))),$$
$$\log M(r) \sim r^l \qquad (r \to +\infty),$$

and hence Theorems 1 and 2 are applicable.

As in all previous examples, there are exceptional values of φ which we cannot treat because our condition (29) is not satisfied. These exceptional values are

$$\varphi = \frac{2\pi}{l} k \qquad (k = 0, 1, 2, l-1).$$

E. *The Functions* sin z *and* cos z

We treat $\sin z$; the treatment of $\cos z$ is almost identical and will be left to the reader

Consider simultaneously

$$f(z) = \frac{\sinh(z^{1/2})}{z^{1/2}} = \sum_{k=0}^{\infty} \frac{z^k}{(2k+1)!} = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\pi^2 n^2}\right), \tag{1.32}$$

and the change of variable

$$t = iz^{1/2}, \tag{1.33}$$

which yields

$$f(-t^2) = \frac{\sin t}{t}.$$

By mth partial sum of the Taylor expansion of $\sin t$ we mean

$$s_m(t) = \sum_{0 \leqslant 2k+1 \leqslant m} \frac{(-1)^k t^{2k+1}}{(2k+1)!}.$$

From (1.32) we deduce that, uniformly as $z \to \infty$,

$$\log f(z) \sim z^{1/2} \qquad (-\pi + \delta \leqslant \arg z \leqslant \pi - \delta, 0 < \delta < \pi),$$

and that

$$\log M(r) \sim r^{1/2} \qquad (r \to +\infty).$$

Hence our function $f(z)$ satisfies the conditions of Theorem 1 and 2 which are therefore applicable.

For

$$\arg z = \varphi = 0$$

the condition (29) is not satisfied and consequently the argument $\varphi = 0$ is exceptional.

As $m \to +\infty$, $m \in \mathfrak{M}_3$ (and $\mathfrak{M}_3$ is suitably chosen) our theorems yield information regarding the behavior of the partial sums

$$\sum_{k=0}^{m-1} \frac{z^k}{(2k+1)!} \qquad (0 < \arg z = \varphi < \pi).$$

In view of (1.33) we may transfer this information to the polynomials

$$s_{2m-1}(t) = \sum_{k=0}^{m-1} \frac{(-1)^k t^{2k+1}}{(2k+1)!}, \tag{1.34}$$

which are sections of the expansion of sin t. The values

$$\arg t = \pm(\pi/2)$$

are exceptional and must be treated by some other method. Because of the uniform convergence of

$$s_{2m-1}(t) \to \sin t \qquad (m \to +\infty),$$

the sections (1.34) will have real zeros (positive as well as negative) whose number tends to ∞ as $m \to +\infty$.

F. *Problems for Further Study*

The results of this paper suggest the possibility of a complete solution of the questions under consideration for all entire functions of finite positive order and of *completely regular growth.*

These functions form a class which was introduced and studied, independently, by Pfluger and Levin. They are characterized by the existence of an angular density of their zeros. (Precise definitions will be found in Levin's book [9].)

For those sectors (with vertex at the origin) free from zeros of $f(z)$, one may expect that some slight modification of Theorem 2 will provide satisfactory answers. The disruption produced by the presence of infinitely many zeros remains to be studied.

It would also be of importance to accurately describe the behavior of the zeros of $s_{m-1}(z)$ along the exceptional rays which appear whenever our condition (29) is not satisfied. That this behavior may be noticeably different from the ordinary one may be seen by studying special functions such as Mittag-Leffler's function $E_{1/\lambda}(z)$. The reader will find in the monograph of Edrei, Saff, and Varga [7] a detailed treatment of $E_{1/\lambda}(z)$ as well as a discussion [7, pp. 5–7] of the relations between the results of the present paper and a conjecture, as yet unsettled, of Saff and Varga.

2. Notational Conventions

By ω we mean a complex quantity such that

$$|\omega| \leqslant 1;$$

ω may be a function of all the variables and parameters of the problem.

By η_j we denote a member, real or complex, of a sequence $\{\eta_j\}$ such that $\eta_j \to 0$ as $j \to +\infty$.

We denote by A a positive absolute constant and by K a positive constant which may depend on all our parameters.

Inequalities such as

$$r > r_0,\ x > x_0,\ m > m_0, \ldots,$$

immediately following some relation mean that it is possible that the relation in question only holds for sufficiently large values of r, x, m,....

The symbols η_j, A, K, r_0, x_0, m_0,..., may have different values in different places.

3. Proof of Theorem 1 and of Corollary 1.1

Throughout the paper the series (1) represents an entire transcendental function so that, as is well known [12; p. 5, ex. 33], $\nu(r)$ is uniquely defined for $r > 0$, and is a non-decreasing, unbounded step function satisfying the relation

$$\log \mu(r) - \log \mu(1) = \int_1^r \frac{\nu(t)}{t}\, dt. \tag{3.1}$$

It is also known that [12; p. 8, ex. 54], for all functions of finite order,

$$\log \mu(r) \sim \log M(r) \qquad (r \to +\infty). \tag{3.2}$$

Assuming the asymptotic behavior stated in (14) we deduce, in view of (3.1) and (3.2),

$$Br^{\lambda}(\log r)^{\alpha} \sim \int_1^r \frac{\nu(t)}{t}\, dt \qquad (r \to +\infty). \tag{3.3}$$

Since $\nu(t)$ is nonnegative and nondecreasing, a straightforward tauberian argument [18; p. 47] yields

$$\nu(r) \sim \lambda B r^{\lambda}(\log r)^{\alpha} \qquad (r \to +\infty), \tag{3.4}$$

which coincides with assertion (15) of Theorem 1.

The points of discontinuity of $v(t)$ $(t>0)$ define a strictly increasing sequence of unbounded, positive numbers. The fact that $v(t)$ is a step function implies the existence of consecutive points of discontinuity of $v(t)$ (we shall denote them by t_m and t'_m) such that

$$v(t)=m \qquad (t_m \leqslant t < t'_m) \tag{3.5}$$

and such that

$$v(t)<m \ (t<t_m), \qquad v(t)>m \ (t'_m \leqslant t).$$

Select for R_m any point of the interval $[t_m, t'_m)$. This yields

$$v(R_m)=m, \tag{3.6}$$

and defines the sequence (16) as m takes on, successively, the values of the terms of $\mathfrak{M}$. The relations (17) now follow by taking $r=R_m$ in (15). We shall say that the sequence (16) is the sequence of critical radii associated with the sequence $\mathfrak{M}$ of critical indices. Although the sequence of critical radii is not uniquely defined we find it convenient to leave it unperturbed throughout the paper.

Let m be the central index appearing in (3.5) and let m^* be the central index immediately preceding m.

It is clearly possible to find τ_m such that

$$t_m - 1 < \tau_m < t_m, \qquad v(\tau_m) = m^*.$$

Then as $m \to +\infty$, $m \in \mathfrak{M}$, we deduce from (15) (since $v(t_m)=m$),

$$m^*/m \to 1 \qquad (m \to +\infty, m \in \mathfrak{M}). \tag{3.7}$$

The asymptotic relation (19) is an obvious consequence of (3.7).

To complete the proof of Theorem 1, we note that, since m is a central index, the definitions (20) and (21) imply

$$a_m \neq 0, \qquad |b_j(m)| = \left|\frac{a_{m+j}}{a_m}\right| R_m^j \leqslant 1 \qquad (j=0, \pm 1, \pm 2,...). \tag{3.8}$$

The inequalities (22) as well as Theorem 1 are thus established.

Corollary 1.1 is almost immediate: the possibility of determining $\mathfrak{M}_2$ so as to satisfy (24) is an obvious consequence of (22) and of the selection principle.

The other assertions of Corollary 1.1 follow from the elements of complex analysis.

There is no difficulty in treating the series

$$\sum_{j=1}^{\infty} b_{-j}(m)\, w^j, \qquad \sum_{j=1}^{\infty} b_{-j} w^j, \tag{3.9}$$

exactly as we have treated $G_m(w)$ and $G(w)$. It may happen that the second series in (3.9) is identically zero; in this case we are unable to derive useful information from the consideration of (3.9).

4. An Elementary Lemma about Uniformly Convergent Sequences of Power Series

The following lemma will be needed somewhat later. It is a simple consequence of uniform convergence and does not depend on the construction (in Section 3) of the functions $G_m(w)$ and $G(w)$.

Lemma 4.1. *Let the functions* $G(w)$, $G_m(w)$ $(m \in \mathfrak{M}_1)$ *be regular for* $|w| < 1$ *and let* $G(w) \not\equiv 0$. *Assume that*

$$G_m(w) \to G(w) \qquad (m \to \infty, m \in \mathfrak{M}_1), \tag{4.1}$$

uniformly on every compact subset of the disk $|w| < 1$.

Then, given ε $(0 < \varepsilon < 1)$, *it is possible to find real quantities*

$$m_0(\varepsilon) > 0, \qquad \kappa(\varepsilon) > 0, \qquad l(\varepsilon) \geqslant 0, \tag{4.2}$$

having the following property: the inequalities

$$|w_0| \leqslant 1 - \varepsilon, \qquad 0 < \rho \leqslant \varepsilon/2, \qquad m \geqslant m_0(\varepsilon), \tag{4.3}$$

imply

$$\max_{\theta} |G_m(w_0 + \rho \varepsilon^{i\theta})| \geqslant \kappa(\varepsilon)\, \rho^l \qquad (m \in \mathfrak{M}, l = l(\varepsilon)). \tag{4.4}$$

Remark. In our application of the above lemma the choice of w_0 will vary with m; the notation w_{0m} instead of w_0 would be more appropriate.

Proof. By assumption $G(w)$ is regular for $|w| < 1$ and does not vanish identically. Hence, for any given ε $(0 < \varepsilon < 1)$, $G(w)$ has at most finitely many zeros, say

$$w_1, w_2, \ldots, w_L \qquad (L \geqslant 0), \tag{4.5}$$

in the disk

$$|w| \leqslant 1 - \varepsilon. \tag{4.6}$$

Let $l = l(\varepsilon)$ be the multiplicity of the zero or zeros of highest multiplicity among (4.5). Take $l = 0$ if $L = 0$, and consider the function of w

$$g(w) = \sup\{|G(w)|, |G'(w)|/1!, |G''(w)|/2!, ..., |G^{(l)}(w)|/l!\}. \quad (4.7)$$

It is clear that $g(w)$ ($|w| \leqslant 1 - \varepsilon$) is a strictly positive, continuous function of w so that we may define

$$\kappa(\varepsilon) = \tfrac{1}{2} \inf_{|w| \leqslant 1 - \varepsilon} g(w) > 0. \quad (4.8)$$

The uniform convergence asserted in (4.1) implies

$$(G_m^{(j)}(w)/j!) \to (G^{(j)}(w)/j!) \qquad (j = 0, 1, 2, ..., l;\ G^{(0)} \equiv G),$$

uniformly in the disk (4.6). Hence there exists $m_0(\varepsilon)$ such that, uniformly in w,

$$\left|\frac{G_m^{(j)}(w)}{j!}\right| \geqslant \left|\frac{G^{(j)}(w)}{j!}\right| - \kappa(\varepsilon)$$
$$(m > m_0(\varepsilon),\ m \in \mathfrak{M},\ j = 0, 1, ..., l). \quad (4.9)$$

From (4.7), (4.8), and (4.9) we conclude that with every w_0 ($|w_0| \leqslant 1 - \varepsilon$), it is possible to associate $s = s(w_0)$ such that

$$g(w_0) = \left|\frac{G^{(s)}(w_0)}{s!}\right| \geqslant 2\kappa(\varepsilon) \qquad (0 \leqslant s \leqslant l(\varepsilon)),$$
$$\left|\frac{G_m^{(s)}(w_0)}{s!}\right| \geqslant \kappa(\varepsilon).$$

By Cauchy's estimate

$$\max_\theta |G_m(w_0 + \rho e^{i\theta})| \geqslant \kappa\rho^s \geqslant \kappa\rho^l \qquad (m \geqslant m_0(\varepsilon),\ m \in \mathfrak{M},\ \rho \leqslant \varepsilon/2),$$

which implies (4.4). This proves Lemma 4.1.

5. The Fundamental Decomposition

In this section we always take $m \in \mathfrak{M}$, where $\mathfrak{M}$ is the sequence (11) of critical indices and

$$R = R_m \qquad (m \in \mathfrak{M}), \qquad \nu(R) = m. \quad (5.1)$$

We consider w as an auxiliary complex variable and start from the obvious decomposition deduced from (1)

$$\frac{f(Rw)}{a_m R^m w^m} = \sum_{j=1}^{m} \frac{a_{m-j}}{a_m} R^{-j} w^{-j} + \sum_{j=0}^{\infty} \frac{a_{m+j}}{a_m} R^j w^j$$

$$(w \neq 0, m \in \mathfrak{M}, R = R_m). \tag{5.2}$$

By definition

$$|a_m| R^m = \mu(R) > 0. \tag{5.3}$$

Introduce the expressions

$$G_m(w) = \sum_{j=0}^{\infty} \frac{a_{m+j}}{a_m} R^j w^j = \sum_{j=0}^{\infty} b_j(m) w^j \qquad (m \in \mathfrak{M}, R = R_m), \tag{5.4}$$

$$Q_m(w) = \frac{s_{m-1}(Rw)}{a_m R^m w^m} = \sum_{j=1}^{m} b_{-j}(m) w^{-j} \qquad (w \neq 0, m \in \mathfrak{M}), \tag{5.5}$$

$$U_m(w) = \frac{f(Rw)}{a_m R^m w^m} \qquad (w \neq 0, m \in \mathfrak{M}). \tag{5.6}$$

From (5.2), (5.4), (5.5), and (5.6) we deduce our fundamental decomposition

$$Q_m(w) = U_m(w) - G_m(w) \qquad (w \neq 0, m \in \mathfrak{M}). \tag{5.7}$$

It may be noted that all the functions $G_m(w)$ are entire so that (5.7) may be used for arbitrarily large values of w.

6. Study of $U_m(te^{i\varphi}w)$

Replace, in (5.6), w by

$$wte^{i\varphi} \qquad (\varphi \text{ real}, 0 < t),$$

where t is a parameter, w is a complex variable such that

$$|w - 1| \leqslant \sin \gamma_1 \qquad (0 < \gamma_1 < \gamma), \tag{6.1}$$

and γ is the quantity in (28).

From (5.6) we thus obtain

$$\log U_m(wte^{i\varphi}) = \log f(Rwte^{i\varphi}) - \log f(Rte^{i\varphi}) + W(wte^{i\varphi}), \tag{6.2}$$

where

$$W(wte^{i\varphi}) = \log f(Rte^{i\varphi}) - \log(a_m R^m) - m \log w - m \log t - im\varphi. \quad (6.3)$$

We now prove

LEMMA 6.1. *Let $f(z)$ and φ satisfy the assumptions of Theorem 2 and consider the auxiliary expression*

$$h(t) = \log |f(Rte^{i\varphi})| - \log \mu(R) - m \log t \qquad (0 < t \leqslant 1). \quad (6.4)$$

Then, if

$$m \in \mathfrak{M}, \qquad m > m_0, \qquad R = R_m, \quad (6.5)$$

it is possible to find $t = t(m)$ such that

$$h(t) = 0, \qquad \tau_0 < t < \tau_1, \quad (6.6)$$

where

$$\tau_0 = \exp\left(-\frac{1+c}{\lambda}\right), \qquad \tau_1 = \exp\left(-\frac{\min(1, c)}{2\lambda + 1}\right). \quad (6.7)$$

Moreover

$$t(m) \to \tau = \tau(\varphi) \qquad (m \to \infty, m \in \mathfrak{M}), \quad (6.8)$$

where τ is the unique solution in (0, 1) *of the equation* (30).

Proof. From our assumptions (26) we conclude that, if r_0 is large enough, $f(z)$ has no zeros in the sector

$$\{z : \varphi_1 \leqslant \arg z \leqslant \varphi_2, |z| \geqslant r_0\}.$$

Hence, if R and m satisfy the conditions (6.5) (and remain fixed), $h(t)$ is a continuous function of t throughout the interval $[\tau_0, \tau_1]$.

To justify our choice of τ_0 and τ_1 given in (6.7), put

$$v(x) = (1 - c)\, x^\lambda - 1 - \lambda \log x,$$

and note that, since $c > 0$ (by (29)), we have

$$v'(x) < 0 \qquad (0 < x \leqslant 1),$$

$$v(x) \geqslant v(\tau_0) > \exp(-(1 + c)) > 0 \qquad (0 < x \leqslant \tau_0). \quad (6.9)$$

For $0<c<1$

$$v(\tau_1)=(1-c)\exp\left(-\frac{c\lambda}{2\lambda+1}\right)-1+\frac{\lambda}{2\lambda+1}c<1-c-1+\frac{c}{c}=-\frac{c}{2}.$$

For $c\geqslant 1$

$$v(\tau_1)=(1-c)\exp\left(-\frac{\lambda}{2\lambda+1}\right)-1+\frac{\lambda}{2\lambda+1}<-\frac{1}{2}.$$

Hence, without case distinction,

$$v(x)\leqslant v(\tau_1)<-\frac{\min(1,c)}{2}<0 \qquad (\tau_1\leqslant x\leqslant 1). \tag{6.10}$$

From (6.9) and (6.10) it follows that, in the interval (0, 1), the equation (30) is satisfied at a single point $x=\tau$, with

$$\tau_0<\tau<\tau_1. \tag{6.11}$$

The relations (6.4), (3.2), (26), (29), (14), and (15) imply

$$h(t)=BR^{\lambda}(\log R)^{\alpha}\,v(t)+o(R^{\lambda}(\log R)^{\alpha}) \qquad (R=R_m\to\infty), \tag{6.12}$$

uniformly for

$$\tau_0\leqslant t\leqslant 1.$$

Hence, by (6.9) and (6.10),

$$h(\tau_0)>0, \qquad h(\tau_1)<0 \qquad (m>m_0).$$

In view of the continuity of $h(t)$, there must exist

$$t=t(m), \qquad \tau_0<t(m)<\tau_1 \qquad (m>m_0) \tag{6.13}$$

such that

$$h(t(m))=0 \qquad (m>m_0). \tag{6.14}$$

There may be several values of t satisfying (6.13) and (6.14); in order to determine a unique $t=t(m)$ we impose the additional condition

$$h(t)<0 \qquad (t(m)<t\leqslant 1,\ m>m_0). \tag{6.15}$$

The possibility of satisfying (6.15) is an obvious consequence of (6.10), (6.11), and (6.12). Moreover, (6.12) and (6.14) imply

$$v(t(m))\to 0=v(\tau) \qquad (m\to\infty,\ m\in\mathfrak{M}),$$

and (6.8) follows. This completes the proof of Lemma 6.1.

7. APPROXIMATION TO $\log f(Rte^{i\varphi}w)$

Put

$$z_0 = Rte^{i\varphi}, \qquad s = Rte^{i\varphi}(w-1), \tag{7.1}$$

where

$$|w-1| < \varepsilon_1 < \frac{1}{2}\sin\gamma_1, \quad 0 < \gamma_1 < \gamma < \frac{\pi}{2}, \quad R = R_m, \quad t = t(m). \tag{7.2}$$

In the above relations $\varepsilon_1 > 0$ is fixed; γ_1 and γ are the quantities which appear in (6.1) and (28).

Let

$$F(s) = \log\left\{\frac{f(z_0+s)}{f(z_0)}\right\} \qquad (F(0)=0), \tag{7.3}$$

and assume that $f(z_0+s)$ is regular and has no zeros in the disk

$$|s| \leqslant 2\varepsilon_1 Rt. \tag{7.4}$$

(This holds as soon as $R_m\tau_0$ is large enough.)

From Cauchy's formula

$$\log f(z_0+s) - \log f(z_0) = \frac{s}{2\pi i}\int_C \frac{F(u)}{u^2}\,du + \frac{s^2}{2\pi i}\int_C \frac{F(u)}{u^2(u-s)}\,du, \tag{7.5}$$

where we may choose for contour of integration

$$C\colon u = 2\varepsilon_1 z_0 e^{i\theta} \qquad (0 \leqslant \theta < 2\pi). \tag{7.6}$$

We now prove

LEMMA 7.1. *Under the assumptions stated as conditions* (26), (27), (28), *and* (29) *of Theorem* 2, *we have, uniformly for all* w *restricted by the inequality* $|w-1| < \varepsilon_1$,

$$\log f(Rt(m)\,e^{i\varphi}w) - \log f(Rt(m)\,e^{i\varphi})$$
$$= \frac{\beta}{B}\tau^\lambda e^{i\varphi\lambda} m(1+\eta_m)(w-1) + \omega K m(w-1)^2 \qquad (m\to\infty, m\in\mathfrak{M}). \tag{7.7}$$

Proof. An inspection of (7.1) and (7.2) shows that when u lies on the contour of integration C, we have

$$|z_0+u| \geqslant |z_0| - 2\varepsilon_1|z_0| = |z_0|(1-2\varepsilon_1) \to +\infty \qquad (m\to\infty, m\in\mathfrak{M}), \tag{7.8}$$

as well as

$$\arg z_0 = \varphi, \qquad |\arg(z_0+u) - \arg z_0| = |\arg(1+2\varepsilon_1 e^{i\theta})| < \gamma_1. \quad (7.9)$$

The relations (7.3), (7.8), and (7.9) enable us to use the asymptotic relation (26) for the evaluation of $F(u)$. We thus obtain, as $m \to +\infty$,

$$\begin{aligned} F(u) &= \beta(z_0+u)^\lambda \left(\log z_0 + \log\left(1+\frac{u}{z_0}\right)\right)^\alpha - \beta z_0^\lambda(\log z_0)^\alpha \\ &\quad + o(|z_0|^\lambda (\log|z_0|)^\alpha \\ &= \beta z_0^\lambda (\log z_0)^\alpha \left\{\left(1+\frac{u}{z_0}\right)^\lambda - 1\right\} + \omega\eta_m |z_0|^\lambda (\log|z_0|)^\alpha, \end{aligned} \quad (7.10)$$

and

$$|F(u)| \leqslant K|z_0|^\lambda (\log|z_0|)^\alpha. \quad (7.11)$$

(We have used our notational conventions in (7.10) and (7.11).)

We now apply (7.10) to evaluate the first integral in (7.5) and (7.11), (7.6), and (7.1) to estimate the second integral; this leads to

$$\begin{aligned} &\log f(z_0+s) - \log f(z_0) \\ &\quad = \lambda\beta z_0^\lambda(\log z_0)^\alpha (w-1) + \omega\eta_m |z_0|^\lambda (\log|z_0|)^\alpha (w-1) \\ &\qquad + \omega K|z_0|^\lambda (\log|z_0|)^\alpha (w-1)^2 \qquad (m\to\infty, m\in\mathfrak{M}, |w-1|<\varepsilon_1), \end{aligned} \quad (7.12)$$

which holds uniformly under the above restrictions.

Writing

$$\xi_m = t(m)\, e^{i\varphi}$$

and using (7.1), (15), and (6.8), we find

$$\begin{aligned} \lambda\beta z_0^\lambda(\log z_0)^\alpha &= \frac{\beta}{B}\xi_m^\lambda m(1+\eta_m) \\ &= \frac{\beta}{B}\tau^\lambda e^{i\varphi\lambda} m(1+\eta_m) \qquad (m\to\infty, m\in\mathfrak{M}). \end{aligned} \quad (7.13)$$

Lemma 7.1 is an obvious consequence of (7.12) and (7.13).

8. Estimates for $U_m(wt(m)\, e^{i\varphi})$

By (6.3) and Lemma 6.1 we find, for $t = t(m)$, $R = R_m$

$$W(wte^{i\varphi}) = i\{\arg f(Rte^{i\varphi}) - \arg a_m - m\varphi\} - m \log w$$
$$= i\chi_m - m \log w. \tag{8.1}$$

It is important to note that the real quantity χ_m is independent of w and that, by a proper selection of the determination of arg a_m, we may assume

$$0 \leqslant \chi_m < 2\pi. \tag{8.2}$$

We now combine (6.2), (7.7), and (8.1) and use the approximation

$$\log w = w - 1 + \omega(w-1)^2 \qquad (|w-1| \leqslant \tfrac{1}{2}).$$

After a few simple reductions we find

$$U_m(t(m)\, e^{i\varphi}w) = \exp\left\{i\chi_m + (w-1)\, m\left(\frac{\beta}{B}\tau^\lambda e^{i\varphi\lambda} - 1\right)\right.$$
$$\left. + \omega\eta_m(w-1)\, m + \omega K(w-1)^2\, m\right\}. \tag{8.3}$$

In (34) we have defined

$$\Lambda = \frac{\beta}{B}\tau^\lambda e^{i\varphi\lambda} - 1.$$

By (29) and (31) we find

$$\tau^\lambda\left(1 - \operatorname{Re}\frac{\beta}{B} e^{i\varphi\lambda}\right) = \tau^\lambda c > 0,$$

and hence

$$1 - \tau^\lambda + \tau^\lambda - \operatorname{Re}\frac{\beta}{B}\tau^\lambda e^{i\varphi\lambda} = \operatorname{Re}\{-\Lambda\} = 1 - \tau^\lambda + \tau^\lambda c > 1 - \tau_1^\lambda > 0. \tag{8.4}$$

This shows that $\Lambda \neq 0$ and we may therefore use, in (8.3), either the change of variable

$$w = 1 + \frac{\zeta T \log m}{\Lambda m} \qquad (T > 0), \tag{8.5}$$

or the change of variable

$$w = 1 + \frac{\zeta}{\Lambda m}. \tag{8.6}$$

We are thus led to

LEMMA 8.1. *Let the assumptions and notations of Lemma 7.1 be unchanged.*

I. *Then, for φ fixed ($\varphi \in [\varphi_1 + \gamma, \varphi_2 - \gamma]$), the functions of ζ defined by*

$$U_m\left(t(m)\, e^{i\varphi}\left(1 + \frac{\zeta}{\Lambda m}\right)\right) \qquad \left(m \in \mathfrak{M}, \Lambda = \frac{\beta}{B}\tau^\lambda e^{i\varphi\lambda} - 1\right) \tag{8.7}$$

are uniformly bounded on every compact set of the ζ-plane.

The quantity Λ is a function of φ such that, for $\varphi \in [\varphi_1 + \gamma, \varphi_2 - \gamma]$,

$$|\Lambda| > 1 - \tau_1^\lambda > 0.$$

If

$$\chi_m \to \chi \qquad (m \to \infty, m \in \mathfrak{M}_3), \tag{8.8}$$

then

$$U_m\left(t(m)\, e^{i\varphi}\left(1 + \frac{\zeta}{\Lambda m}\right)\right) \to \exp(i\chi + \zeta) \qquad (\chi \text{ real}). \tag{8.9}$$

II. *If $T > 0$ is given and $|\zeta| \leqslant 1$, then*

$$U_m\left(t(m)\, e^{i\varphi}\left(1 + \frac{\zeta T \log m}{\Lambda m}\right)\right)$$
$$= \exp\left(i\chi_m + \zeta T \log m\left(1 + \omega\frac{\eta_m}{\Lambda}\right) + \omega K\frac{T^2(\log m)^2}{\Lambda^2 m}\right). \tag{8.10}$$

Proof. Assertion I of the lemma is an obvious consequence of (8.6), (8.3) and of our definition of Λ.

Similarly, assertion II follows from (8.5) and (8.3).

9. PROOF OF ASSERTIONS I, II, AND III OF THEOREM 2

In Lemma 6.1 we have established the existence of $t(m)$; we have shown that $t(m)$ satisfies the limit relation (32) and that the limit $\tau(\varphi)$ lies in (τ_0, τ_1). We have also shown that (30) is satisfied for $x = \tau(\varphi)$.

The assertion $\Lambda \neq 0$, which is stated in (34), has been established in (8.4). Hence the consideration of (8.5), (8.6), (8.9), and (8.10), as well as the introduction of $X_m(\zeta)$ in (33), offers no difficulties.

From (5.7) we deduce

$$Q_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right)$$
$$= U_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right) - G_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right). \tag{9.1}$$

By Lemma 8.1 we know that, on every compact set of the ζ-plane

$$U_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right)$$

is uniformly bounded provided

$$\varphi_1+\gamma \leqslant \varphi \leqslant \varphi_2-\gamma, \qquad m \in \mathfrak{M}. \tag{9.2}$$

If $m \in \mathfrak{M}$ is large enough, we have, by (31) and (32),

$$t(m)\left|1+\frac{\zeta}{\Lambda m}\right| < \frac{1+\tau_1}{2} < 1$$

so that, by Corollary 1.1,

$$\left|G_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right)\right| < \frac{2}{1-\tau_1}. \tag{9.3}$$

Hence (9.1) and (9.3) imply the uniform boundedness of

$$Q_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right).$$

Returning to (5.5) we see that this implies the boundedness of $X_m(\zeta)$ in (33).

Since the asymptotic relation (35) was established in (19) we see that the proofs of assertions I and II of Theorem 2 are now complete.

To prove assertion III of Theorem 2 we note that, by suitably choosing $\mathfrak{M}_2 \subset \mathfrak{M}_1$, Corollary 1.1 yields

$$G_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right) \to G(\xi) \qquad (m \to \infty,\ m \in \mathfrak{M}_2). \tag{9.4}$$

By Lemma 8.1 an adequate choice of $\mathfrak{M}_3 \subset \mathfrak{M}_2$ implies (8.8) and (8.9); hence (9.1) and (9.4) imply

$$Q_m\left(t(m)\,e^{i\varphi}\left(1+\frac{\zeta}{\Lambda m}\right)\right) \to \exp(i\chi+\zeta)-G(\xi) \qquad (m\to\infty,\, m\in\mathfrak{M}_3). \quad (9.5)$$

The limit relation (37) follows from (9.5) and (5.5) and the proof of Theorem 2 is complete.

10. Proof of Corollary 2.1

We start from the limit relation (37). The zeros of the limit function $\tilde{X}(\zeta)$ are given by the formula

$$\zeta_k = \log G(\xi) - i\chi + 2k\pi i \qquad (k=0,\ \pm 1,\ \pm 2,\ \pm 3,...).$$

It is immediately seen that the inequalities

$$|\zeta_k| \leqslant r \tag{10.1}$$

are satisfied by no more than

$$\frac{r}{\pi}+\frac{|\tilde{\gamma}|}{\pi}+1 \qquad (\tilde{\gamma}=\log G(\xi)-i\chi) \tag{10.2}$$

values of k.

Similarly there are at least

$$\frac{r}{\pi}-\frac{|\tilde{\gamma}|}{\pi}+1 \tag{10.3}$$

acceptable values of k for which (10.1) holds.

In view of Hurwitz' theorem, the relations (33) and (37) show that, to the zero ζ_k ($|\zeta_k|<r$) of $\tilde{X}(\zeta)$ (this zero is necessarily simple), there corresponds a simple zero ζ_{mk} of $s_{m-1}(z)$, given by the relation

$$\zeta_{mk} = R_m t(m)\,e^{i\varphi} + R_m t(m)\,e^{i\varphi}\,\frac{(\zeta_k+\eta_m)}{\Lambda m}.$$

Hence taking into account the upper and lower bounds (10.2) and (10.3), of acceptable values of k, we immediately verify Corollary 2.1.

The validity of our arguments requires that $G(\xi)\neq 0$. Should we have $G(\xi)=0$ some modifications are in order; they will be considered in Theorem 3.

11. Proof of Theorem 3

Since $\mathfrak{M}_3$ has been chosen, the limit function

$$G(w) = \lim_{\substack{m \to \infty \\ m \in \mathfrak{M}_3}} G_m(w) \tag{11.1}$$

is well determined. By assertions I and II of Theorem 2 we have

$$t(m) \to \tau = \tau(\varphi) \qquad (m \to \infty,\ m \in \mathfrak{M}_3)$$

with

$$0 < \tau_0 < \tau(\varphi) < \tau_1 < 1. \tag{11.2}$$

Hence

$$0 < t(m)\left(1 + \frac{K \log m}{m}\right) < \tau_1^{1/2} \qquad (m > m_0,\ m \in \mathfrak{M}_3). \tag{11.3}$$

We now apply Lemma 4.1 to the functions $G(w)$, $G_m(w)$ $(m \in \mathfrak{M}_3)$ with

$$\varepsilon = \frac{1 - \tau_1^{1/2}}{2}. \tag{11.4}$$

The lemma determines the three real quantities

$$m_0(\varepsilon) > 0, \qquad \kappa = \kappa(\varepsilon) > 0, \qquad l = l(\varepsilon) \geqslant 0. \tag{11.5}$$

We assume, in addition, that $\kappa \leqslant 1$ (an inspection of (4.4) shows that this is always possible).

Take

$$\rho = \frac{Tt(m)}{8\,|\Lambda|}\left(\frac{\log m}{m}\right) \qquad (m > m_0), \tag{11.6}$$

where

$$T = 2\,\frac{l+1}{\sigma}, \tag{11.7}$$

and $\sigma > 0$ is an absolute constant to be introduced below in (11.22). We shall find

$$\sigma < (1/50) \tag{11.8}$$

so that

$$\frac{T}{17} > \frac{100(l+1)}{17} > 2(l+1). \tag{11.9}$$

Put

$$\zeta_{0m} = -\frac{1}{4} + \frac{e^{i\theta}}{8} \qquad (\theta \text{ real}), \tag{11.10}$$

and choose $\theta = \theta(m)$ such that

$$\left| G_m \left(t(m)\, e^{i\varphi} \left(1 + \frac{\zeta_{0m} T \log m}{\Lambda m} \right) \right) \right| \geqslant \kappa \left(\frac{T t(m) \log m}{8 \,|\Lambda|\, m} \right)^l \qquad (m > m_0). \tag{11.11}$$

That this is possible is a consequence of Lemma 4.1, (11.1), and (11.2).

We propose to apply Lemma A to the auxiliary function of ζ:

$$\begin{aligned} \phi_m(\zeta) &= 2\kappa^{-1} m^l Q_m \left(t(m)\, e^{i\varphi} \left(1 + \frac{\zeta T \log m}{\Lambda m} \right) \right) \\ &= 2\kappa^{-1} m^l \left\{ U_m \left(t(m)\, e^{i\varphi} \left(1 + \frac{\zeta T \log m}{\Lambda m} \right) \right) \right. \\ &\quad \left. - G_m \left(t(m)\, e^{i\varphi} \left(1 + \frac{\zeta T \log m}{\Lambda m} \right) \right) \right\} \qquad (m > m_0,\ m \in \mathfrak{M}_3). \end{aligned} \tag{11.12}$$

An inspection of (5.5) reveals that $\phi_m(\zeta)$ has a multiple pole at

$$\zeta = -\frac{\Lambda m}{T \log m}.$$

This need not prevent us from applying Lemma A to $\phi_m(\zeta)$ $(m > m_0)$ because, for m large enough, the pole in question lies outside the disk $|\zeta| \leqslant 1$.

By (11.10) we have

$$\operatorname{Re} \zeta \leqslant -\tfrac{1}{16}, \qquad |\zeta| \leqslant \tfrac{7}{16} \tag{11.13}$$

throughout the disk

$$D_m = \{\zeta : |\zeta - \zeta_{0m}| \leqslant \tfrac{1}{16}\}. \tag{11.14}$$

In view of (11.13), (11.9), and (8.10) we find

$$\left| U_m \left(t(m)\, e^{i\varphi} \left(1 + \frac{\zeta T \log m}{\Lambda m} \right) \right) \right| < m^{-T/17} < m^{-2l-2} \qquad (m > m_0,\ \zeta \in D_m). \tag{11.15}$$

By (24) and (11.3) we also have

$$\left| G_m\left(t(m)\, e^{i\varphi}\left(1+\frac{\zeta T\log m}{Am}\right)\right)\right| \leqslant \frac{1}{1-\tau_1^{1/2}} \qquad (m>m_0,\, |\zeta|\leqslant 1). \tag{11.16}$$

Using (11.15), (11.16), and (11.11) in (11.12) we find

$$|\phi_m(\zeta)| < \frac{3\kappa^{-1}}{1-\tau_1^{1/2}}\, m^l \qquad (m>m_0,\, \zeta\in D_m), \tag{11.17}$$

$$|\phi_m(\zeta_{0m})| > 1 \qquad (m>m_0). \tag{11.18}$$

For any fixed value of $T>0$, the inequalities (11.17) and (11.18) hold for sufficiently large values of m_0.

From (11.12), (11.16), and (8.10) we deduce

$$\max_{|\zeta|=1} |\phi_m(\zeta)| = m^{T+l+\eta_m}, \tag{11.19}$$

$$\max_{|\zeta|=1/2} |\phi_m(\zeta)| = m^{T/2+l+\eta_m}. \tag{11.20}$$

In our application of Lemma A to the function $\phi_m(\zeta)$ we select the quantities ζ_0, σ_1, σ_2, and h in (46) to be

$$\zeta_0 = \zeta_{0m}, \qquad \sigma_1 = \tfrac{1}{16}, \qquad \sigma_2 = \tfrac{1}{2}, \qquad h = \tfrac{1}{3}. \tag{11.21}$$

Then, the auxiliary quantities σ and h_1 in (49) are positive absolute constants which satisfy the inequalities

$$h_1 < 1, \qquad \sigma < (1/50). \tag{11.22}$$

The disk D_m in (11.14) has radius $\sigma_1 = (1/16)$ and center ζ_{0m}. In view of (11.10) and (11.18) the conditions corresponding to (45) are certainly satisfied for $\phi = \phi_m$; moreover, by (11.21)

$$|\zeta_0| \leqslant \tfrac{3}{8}, \qquad |\zeta_0| + \sigma_1 \leqslant \tfrac{7}{16} < \sigma_2 = \tfrac{1}{2},$$

so that the conditions corresponding to (46) are also satisfied. The conditions (11.19), (11.20), and (11.17) imply

$$H_1 = (T+l+\eta_m)\log m,$$

$$H_2 = \left(\frac{T}{2}+l+\eta_m\right)\log m, \qquad H_0 < l\log m + \log\left(\frac{3\kappa^{-1}}{1-\tau_1^{1/2}}\right). \tag{11.23}$$

Hence, in view of (11.23)

$$\frac{H_2}{H_1} > \frac{1}{2} \frac{T+l+\eta_m}{T+l+\eta_m} > \frac{1}{3} = h \qquad (m > m_0);$$

the condition corresponding to (48) is thus satisfied. Similarly, using (11.23) and (11.7)

$$\frac{H_0}{H_1} < \frac{l + (1/\log m) \log(3\kappa^{-1}/(1-\tau_1^{1/2}))}{T+l+\eta_m} < \sigma \qquad (m > m_0).$$

Lemma A now asserts that $\phi_m(\zeta)$ has no fewer than

$$\frac{\sigma(T+l+\eta_m)}{6 \log 2} \log m > T_1 \log m \qquad (m > m_0),$$

zeros in the disk $|\zeta| \leqslant 1$. In view of (11.7) we see that the choice

$$T_1 = \frac{l+1}{6 \log 2}$$

is acceptable. Hence T_1, like l and T, is independent of φ. From (5.5) and (11.12) it follows that the number of zeros of $\phi_m(\zeta)$ in $|\zeta| \leqslant 1$ coincides with the number of zeros of $s_{m-1}(z)$ in the disk defined by (43). The proof of Theorem 3 is now complete. The quantity $|A|$ in (43) still depends on φ. Total uniformity with respect to φ may be achieved by replacing in (43), $|A|$ by the lower bound $1-\tau_1^{\lambda}$ deduced from (8.4).

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